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Massless Thirring model in curved space: Thermal states and conformal anomaly

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The massless Thirring model of a self-interacting fermion field in a curved two-dimensional background spacetime is considered. The exact operator solution for the fields and the equation for the two-point function are given and used to examine the radiation emitted by a two-dimensional black hole. The radiation is found to be thermal in nature, confirming general predictions to this effect. We compute the particle spectrum of the Thirring fermions at finite temperature in Minkowski space and point out errors in a previous attempt at this calculation. Finally we calculate the vacuum expectation value of the stress tensor in an arbitrary two-dimensional spacetime by exploiting the connection between the thermal Hawking radiation from a black hole and the value of the so-called conformal trace anomaly. The latter is also computed quite independently, using dimensional regularization, from general considerations of the renormalization group. We find the stress tensor to be identical to the free-fermion case. Some discussion of the significance of this work for the second law of thermodynamics and black-hole entropy is given.

I. INTRODUCTION

Recent activity on quantum field theory in a background (classical) gravitational field has concentrated almost entirely on free fields obeying linear wave equations. (For a review, see Parker.[1]) Great interest is attached to whether the essential results on this activity will survive when interactions are included; for example, will a black hole still emit *thermal* radiation? Moreover, might the presence of interactions give rise to quantum effects, such as particle production and vacuum polarization, that overwhelm the free field contribution in magnitude?

Unfortunately the development of interacting quantum field theory in curved spacetime is fraught with fundamental mathematical and conceptual problems and only a handful of results have so far been obtained (see, for example, the work of Drummond and collaborators[2-4]). Therefore, as an attempt to answer the above questions, we have chosen to investigate an idealized and specialized exactly soluble model of a self-interacting quantum field, rather than seek partial results for a more general system. The model is a field ϕ of massless fermions propagating in two-dimensional spacetime (z, t): the massless Thirring model.[5] It has been much studied in a Minkowski space context and is known to be equivalent to a free boson field theory.[6]

The generalization to curved spacetime is straightforward because the model is invariant under conformal transformations, and all two-dimensional spacetimes are conformally related to Minkowski space. Hence we easily transform the known flat-space solutions to curved space. This procedure was originally carried out by Scarf,[7] who made a limited study of particle production. Our interest centers on two aspects of the model: The production of particles by the gravitational field (curved space) and the evaluation of the stress tensor $T_{\mu\nu}$ in certain quantum states.

We only investigate particle production in one situation of particular interest and simplicity: The evaporating black hole discovered by Hawking.[8] It is generally believed that the thermal character of black-hole radiation is fundamentally connected with the nature of the event horizon, the information loss across it, and the universality of the second law of thermodynamics. However, there exists no general proof that the radiation from a black hole must be thermal. In the free-field case the result apparently follows fortuitously from the properties of the special functions which appear in the Bogolubov transformation between the in and out states associated with a collapsing star. It is widely hoped that the result is, in fact, not a mathematical accident, but has deep significance, relating the structure of spacetime to the laws of thermodynamics.

With this fundamental speculation in mind, we were gratified to discover that the presence of the self-interaction term in the Thirring model does not destroy the thermal nature of the Hawking radiation. Previously, Gibbons and Perry[9] argued on general grounds that the thermal result would survive interactions, so our work confirms their conjecture, in this model at least.

The consequences for physics of nonthermal black-hole emission are catastrophic. Suppose a black hole is immersed in a bath of thermal radiation, the temperature of which is adjusted so that the rate of energy absorption by the hole exactly balances its (nonthermal) emission. Then the mass, hence area, of the hole will remain unchanged. But the area is proportional to the *entropy* of the hole, so this too remains constant. The net effect is thus for the hole to absorb high-entropy (thermal) radiation and emit low-entropy (nonthermal) radiation while its own entropy is unchanged. It follows that the total entropy of the system steadily diminishes, in violation of the second law of thermodynamics.

It comes as something of a surprise that interacting quantum field theory in black-hole spacetimes should contrive to save the second law of thermodynamics, for this law is apparently nowhere built into the laws of quantum field theory. Other examples of quantum field theory contriving to save the second law have been discovered by Ford.[10] This remarkable consistency between such dissimilar branches of physics strongly suggests that the second law of thermodynamics is not merely fortuitously obeyed by curved-space quantum fields, but that there indeed exists a fundamental connection of some sort between gravity, quantum theory, and entropy.

Our study of black-hole evaporation leads us to consider thermal states for the Thirring model. We find that previous work on this subject by Deo and Kumar[11] contains several errors, and we believe that their results are incorrect. (In fact, they imply non-thermal black-hole emission.)

We treat the thermal states and black-hole system in Sec. III, after a brief review of the Thirring model in Sec. II. In Sec. IV we investigate the stress tensor $T_{\mu\nu}$. Because of the conformal triviality it may be shown (e.g. Davies[12]) that the form of $T_{\mu\nu}$ for any two-dimensional spacetime is uniquely specified by the quantum state together with the so-called anomalous trace $T^{\mu}{}_{\mu}$. This object is normally computed by regularization of the (formally divergent) stress tensor. While this is straightforward for free fields (see, for example, Davies and Fulling,[13] Davies and Unruh,[14] Davies,[15] Bunch, Christensen, and Fulling[16]) the case of interacting fields is more complicated[2] because it involves both a renormalization of fields and perhaps coupling constants and masses as well, together with a "vacuum subtraction," i.e., a renormalization of coupling constants in the gravitational field equations. These *two* types of renormalization, one due to the quantum field interaction, the other due to the coupling of the quantum field to the classical gravitational field, get tangled together, and existing prescriptions for regularization need to be extended to cover these more complicated cases.

Fortunately, however, we do not need to resort to brute-force regularization in this case. Instead we use an elegant result of Christensen and Fulling[17] that relates the conformal anomaly to the asymptotic Hawking flux. Because the spacetime is asymptotically flat, and because the Hawking radiation is thermal, we may simply use the results about thermal radiation *in Minkowski space* to evaluate the anomalous trace in *curved space*. This in turn enables the entire stress tensor in *any* curved space (not just the black-hole case) to be constructed.

In Sec. VI we relate the results of this work to that of Drummond and Shore,[3] who have proposed a method of determining the trace anomaly for any conformally invariant field theory in an arbitrary background, while in the final section we mention briefly possible areas for future research on interacting field in curved spacetimes.

II. THE MODEL

We work in two-dimensional spacetime with coordinates $x \equiv (t, z)$ and related null coordinates $u \equiv t - z$, $v \equiv t + z$. The metric has manifestly conformally flat form

$$ds^2 = C(u, v)\, du dv\,. \tag{2.1}$$

The Thirring field ϕ has Lagrangian density

$$\mathcal{L} = \tfrac{1}{2} i(\bar{\phi}\gamma^{\mu}\phi_{;\mu} - \bar{\phi}_{;\mu}\gamma^{\mu}\phi) + \lambda J^{\mu} J_{\mu} \tag{2.2}$$

and field equation

$$i\gamma^{\mu}\phi_{;\mu} + \lambda J^{\mu}\gamma_{\mu}\phi = 0\,, \tag{2.3}$$

where λ is a coupling constant and J^{μ} is the usual fermion current, given formally by

$$J^{\mu} = \bar{\phi}\gamma^{\mu}\phi\,. \tag{2.4}$$

In curved space, or in flat space with non-Minkowskian coordinates, the γ matrices are related to their flat-space, Minkowski counterparts by

$$\gamma^{u} = C^{-1/2}\tilde{\gamma}^{u}\,, \quad \gamma^{v} = C^{-1/2}\tilde{\gamma}^{v}\,, \tag{2.5}$$

and we choose the representation

$$\tilde{\gamma}^{u} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad \tilde{\gamma}^{v} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{2.6}$$

The field equation (2.3) may be rewritten[7]

$$iC^{-3/4}\tilde{\gamma}^{u}(C^{1/4}\phi)_{,u} + \lambda \tilde{J}^{\mu}\tilde{\gamma}_{\mu}\phi = 0\,, \tag{2.7}$$

which has immediate solution

$$\phi(u, v) = C^{-1/4}(u, v)\tilde{\phi}(u, v) \tag{2.8}$$

in terms of the flat-space solution $\tilde{\phi}$. The latter is found in the work of Klaiber,[18] who gives a detailed review of the Thirring model in flat space. We summarize Klaiber's solution in Appendix A.

In our calculations of the thermal radiation and the stress tensor, we do not use the solution (A1) directly, but work instead with the two-point function found by Johnson[19] and given in a more general form by Klaiber.[18] In Klaiber's notation, for flat spacetime and point separation Δx,

$$\langle 0|\tilde{\phi}(x'')\tilde{\phi}^*(x')|0\rangle = -ie^{-i(a+b)D^-(\Delta x)}[\tilde{S}^-(\Delta x)\tilde{\gamma}_0], \tag{2.9}$$

where, for Johnson's solution [noting the minor error in Klaiber's Eq. (X.16)],

$$a = -\frac{\lambda}{1+\lambda/2\pi}, \quad b = \frac{\lambda}{1-\lambda/2\pi}, \quad \tilde{\gamma}_0 = \tfrac{1}{2}(\tilde{\gamma}_u + \tilde{\gamma}_v).$$

The function D^- is the usual homogeneous Green's function for a massless scalar field,

$$D^-(x) = \frac{1}{4\pi i}\ln[-m^2(\Delta u - i\epsilon)(\Delta v - i\epsilon)], \tag{2.10}$$

with m an infrared cutoff mass, and $\tilde{S}^-$ is the usual homogeneous free-field fermion Green's function. (The tilde denotes that we are dealing with the *flat*-spacetime quantity.) Thus (2.9) may be written (leaving implicit the quantities $i\epsilon$) as

$$\langle 0|\tilde{\phi}(x'')\tilde{\phi}^*(x')|0\rangle = \frac{m}{2\pi i}e^{-i\pi c}\begin{bmatrix} (m\Delta u)^{-c}(m\Delta v)^{-c-1} & 0 \\ 0 & (m\Delta u)^{-c-1}(m\Delta v)^{-c}\end{bmatrix}, \tag{2.11}$$

where $c = (a+b)/4\pi$.

In arriving at (2.9) and hence (2.11) an infinite wave-function renormalization has been carried out. For a discussion of this see Thirring's original paper[5] and the detailed treatment of Mueller and Trueman.[20] This is the only renormalization necessary; in particular, there is no renormalization of the coupling constant λ.

To generalize Eq. (2.11) to curved space, each $\tilde{\phi}$ operator, according to (2.8), is weighted by a factor $C^{-1/4}$. In addition, the spinor at the spacetime point x' will suffer a transformation if transported to x''. We assume that x', x'' lie along a (non-null) geodesic, and insert a parallel-transport operator P to act on the spinor at x':

$$C^{-1/4}(x'')C^{-1/4}(x')\langle 0|\tilde{\phi}(x'')(P\tilde{\phi}(x'))^*|0\rangle. \tag{2.12}$$

Using (2.9) this may be rewritten as

$$-ie^{-i(a+b)D^-(\Delta x)}[S^-(\Delta x)\tilde{\gamma}_0], \tag{2.13}$$

where the *free*-field propagator for curved space is given by

$$S^-(\Delta x) = C^{-1/4}(x'')C^{-1/4}(x')\langle 0|\tilde{\psi}(x'')(P\tilde{\psi}(x'))^*\tilde{\gamma}_0|0\rangle, \tag{2.14}$$

$\tilde{\psi}$ being the free-fermion field in flat spacetime.

The free-field quantity $S^-(\Delta x)$ has been studied in detail by Davies and Unruh[14] and Davies.[15] In particular, they give the explicit form for the parallel-transport operator P for point separation Δx as

$$P = A\tilde{\gamma}_v\tilde{\gamma}_u + A^{-1}\tilde{\gamma}_u\tilde{\gamma}_v, \tag{2.15}$$

where $A = [C(x')/C(x'')]^{1/4}\,U^{1/2}$ and U is the ratio of the (normalized) tangent vectors $t^{u'}/t^{u''}$ to the geodesic at x', x''. From the normalization condition $Ct^ut^v = 1$, we could also write $V \equiv t^{v'}/t^{v''} = U^{-1}$.

III. THERMAL STATES AND BLACK-HOLE EVAPORATION

To consider the Thirring model at a finite temperature in Minkowski space one studies the two-point (thermal Green's) function

$$\mathrm{Tr}[e^{-\beta H}\tilde{\phi}(x'')\tilde{\phi}^*(x')]/\mathrm{Tr}e^{-\beta H}, \tag{3.1}$$

where the field product is averaged over a grand canonical ensemble.

H is the total Hamiltonian and $\beta = (kT)^{-1}$ for a temperature T. We treat only the case of zero chemical potential.

The expression (3.1) has been investigated by Deo and Kumar[11] and Dubin.[21] Deo and Kumar calculate the particle and energy spectrum for the Thirring model from (3.1) using a method which is only applicable for fields which have c-number unequal-time anticommutators.[22,23] This certainly is not the case for interacting fermion fields in general and for the Thirring model in particular,[24] thereby invalidating the results of Deo and Kumar. The work of Dubin does not suffer from these shortcomings, but treats a more general class of models, so that some work is needed to extract the information relevant to the considerations here. Dubin's equation (6.15) simplifies when restricted to the Thirring model [his $B_\nu(m_{J+};x_j - y_k)$ terms vanish] and to two-point functions (his product $\Pi_{j<k}$ does not contribute). We are left with the following for the thermal Green's function:

$$-i\exp[-4\pi i c\, \mathrm{B}_\nu(0,\Delta x)]\sum_{n=-\infty}^{\infty}(-1)^n \tilde{S}^-(\Delta u+in\beta,\Delta v+in\beta)\,, \tag{3.2}$$

where $\tilde{S}^-$ is our free-field two-point function for flat spacetime and B_ν is given by a sum of two expressions from Dubin's equations (2.33) and (2.34a). The former may be evaluated (for zero chemical potential) to give

$$\frac{i}{4\pi}\int_0^\infty \omega^{-1}(e^{\beta\omega}-1)^{-1}(e^{-i\omega\Delta u}+e^{-i\omega\Delta v})d\omega\,. \tag{3.3}$$

This expression is infrared divergent, so we must introduce a cutoff $me^{-\gamma}$ (γ is Euler's constant). The integral in (3.3) may then be computed to yield

$$-\frac{i}{4\pi}\sum_{n=1}^{\infty}\ln[m^2(\beta n+i\Delta u)(\beta n+i\Delta v)]\,. \tag{3.4}$$

In arriving at (3.4) we have omitted a term which vanishes in the limit $m\to 0$. The other contribution to B_ν turns out to be

$$-\frac{i}{4\pi}\sum_{n=0}^{\infty}\ln[m^2(\beta n-i\Delta u)(\beta n-i\Delta v)]\,. \tag{3.5}$$

Consequently, we find Dubin's expression for the Thirring-model thermal Green's function to be

$$-i\exp\left\{-c\sum_{r=-\infty}^{\infty}\ln[-m^2(\Delta u+i\beta r)(\Delta v+i\beta r)]\right\}\sum_{n=-\infty}^{\infty}(-1)^n\tilde{S}^-(\Delta u+in\beta,\Delta v+in\beta)\,. \tag{3.6}$$

Expression (3.6) has the expected form of a thermal Green's function, being antiperiodic in imaginary time with period $2\pi\beta^{-1}$. Note that in the free-field limit ($c\to 0$) the expression reduces to an infinite (alternating) sum of images of the vacuum Green's function $\tilde{S}^-(\Delta u,\Delta v)$. This form follows automatically from the structure of the anticommutation relations, provided the anticommutators are c numbers. As was pointed out earlier, for the Thirring model this is not so, and the full expression (3.6) is *not* a simple image sum.

We now consider the relation between (3.6) and the radiation emitted by a model black hole. In two dimensions, the analog of Schwarzschild's spacetime has the metric (2.1) with $C=(1-2M/z)$ restricted to the region $z>0$. An alternative coordinate system which is nonsingular at $z=2M$ is due to Kruskal,

$$\begin{aligned}\bar{u}&=-e^{-(t-z^*)/4M}\xrightarrow[z\to\infty]{}-e^{-u/4M}\,,\\ \bar{v}&=e^{(t+z^*)/4M}\xrightarrow[z\to\infty]{}e^{v/4M}\,,\end{aligned} \tag{3.7}$$

with $dz^*=C^{-1}dz$, when the conformal factor C is transformed to $\bar{C}=16M^2e^{z^*/2M}C$. In these expressions the constant M plays the role of the black-hole mass.

Associated with each coordinate system is a set of quantum states, and in particular a vacuum state. Thus each system has an associated two-point function given in terms of (2.11) with either u, v or $\bar{u}$, $\bar{v}$ on the right-hand side. If we wish to model the quantum field outside a black hole in equilibrium with the surrounding environment, it is necessary to use the *Kruskal* vacuum, which we denote by $|\bar{0}\rangle$ (see, for example, the review article by Davies[25]). Inspection of (3.7) shows that the $\bar{u}$, $\bar{v}$ coordinates are periodic in imaginary time $t=\frac{1}{2}(v+u)$ with period $8\pi M$, which already suggests that the Kruskal vacuum corresponds to a *thermal* state as far as the states associated with the Schwarzschild (u,v) coordinates are concerned. The significance of this is that far from the hole ($z\to\infty$) spacetime is flat and the Schwarzschild metric reduces to $du\,dv$, i.e., the standard Minkowski metric. (This is *not* true of the Kruskal states and metric $\bar{C}\,d\bar{u}\,d\bar{v}$.) Thus the Schwarzschild states reduce in this region to the standard quantum states of conventional flat-space field theory.

We may verify this conjecture, explicitly computing (2.13) in $\bar{u}$, $\bar{v}$ coordinates, using the transformation (3.7) to convert to u,v coordinates, and considering the asymptotic (flat-space) region $z\to\infty$. To accomplish this we first consider the free-field part $S^-(\Delta x)$, given by (2.14). The expectation value may be evaluated as a standard mode integral[15] to yield, in terms of barred coordinates,

$$S^-(\Delta\bar{u},\Delta\bar{v})=\frac{1}{4\pi}\left(\frac{\bar{U}^{1/2}\gamma^{\bar{u}''}}{\Delta\bar{u}}+\frac{\bar{V}^{1/2}\gamma^{\bar{v}''}}{\Delta\bar{v}}\right). \tag{3.8}$$

Noting that in the asymptotically flat region far from the black hole

$$t^{\bar{u}}=\frac{d\bar{u}}{du}t^u=(4M)^{-1}e^{-u/4M}t^u,$$

$$t^{\bar{v}}=\frac{d\bar{v}}{dv}t^v=(4M)^{-1}e^{v/4M}t^v,$$

we find, from the transformation Eq. (3.7),

$$\frac{\bar{U}^{1/2}}{\Delta\bar{u}}=\frac{e^{u''/4M}}{2\sinh(\Delta u/8M)}$$

$$=4Me^{u''/4M}\sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n}{\Delta u+8\pi Min}\right], \tag{3.9}$$

$$\frac{\bar{V}^{1/2}}{\Delta\bar{v}}=\frac{e^{-v''/4M}}{2\sinh(\Delta v/8M)}$$

$$=4Me^{-v''/4M}\sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n}{\Delta v+8\pi Min}\right]. \tag{3.10}$$

Using

$$\gamma^{\bar{u}''}=\frac{d\bar{u}''}{du''}\gamma^{u''}=(4M)^{-1}e^{-u''/4M}\gamma^{u''},$$

$$\gamma^{\bar{v}''}=\frac{d\bar{v}''}{dv''}\gamma^{v''}=(4M)^{-1}e^{v''/4M}\gamma^{v''},$$

one obtains from (3.8), (3.9), and (3.10)

$$S^-(\Delta\bar{u},\Delta\bar{v})=\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{\tilde{\gamma}^{u''}}{\Delta u+8\pi Min}+\frac{\tilde{\gamma}^{v''}}{\Delta v+8\pi Min}\right)$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n\tilde{S}^-(\Delta u+8\pi Min,\Delta v+8\pi Min) \tag{3.11}$$

(note that $\gamma^u \underset{z\to\infty}{\rightarrow} \tilde{\gamma}^u$, etc.), which is a thermal Green's function with imaginary time image periodicity $(4M)^{-1}$, corresponding to a temperature $(8\pi kM)^{-1}$. This is precisely the temperature for a black hole of mass M originally calculated by Hawking[8] by evaluating the Bogolubov coefficients between in and out free-field states in the asymptotically flat region around an imploding star.

To extend the treatment to the Thirring case it is only necessary to return to (2.13), in $\bar{u},\bar{v}$ coordinates, insert (3.11) for S^-, and evaluate the exponent $-i(a+b)D^-(\Delta\bar{u},\Delta\bar{v})$ in terms of the u,v coordinates as $z\to\infty$. From Eq. (2.10) the latter is

$$-c\ln(-\bar{m}^2\Delta\bar{u}\Delta\bar{v}),$$

where $\bar{m}$ is an infrared cutoff parameter in the barred coordinate system. Using (3.7) this becomes

$$-c\ln[-64m^2M^2\sinh(\Delta u/8M)\sinh(\Delta v/8M)], \tag{3.12}$$

where m is the transformed cutoff parameter appropriate for the u,v coordinate system

$$m=\frac{\bar{m}}{4M}e^{(z''+z')/8M}.$$

(It is necessary that the cutoff *proper* length for each field be the same in the two systems. Note also that we have $\hbar=c=G=1$.)

Using the well-known identity

$$\sinh x=x\prod_{r=1}^{\infty}\left(1+\frac{x^2}{r^2\pi^2}\right)$$

expression (3.12) may be written as

$$-c\ln\left\{\prod_{r=-\infty}^{\infty}[-m^2(\Delta u+8\pi Mir)(\Delta v+8\pi Mir)]\Big/\prod_{r=1}^{\infty}(8\pi rmM)^4\right\},$$

so that finally one obtains for the two-point function (2.14) (evaluated in the $\bar{u},\bar{v}$ coordinate system) precisely Dubin's expression (3.6) with $\beta=8\pi M$ corresponding to $T=(8\pi kM)^{-1}$, except for the infinite factor

$$\prod_{r=1}^{\infty}(8\pi rmM)^{4c}.$$

This factor is missing from Dubin's expression because in arriving at (3.6) we omitted the terms which vanish in the limit $m\to 0$. It is an unimportant factor anyway as it can be absorbed in the wave-function renormalization.

The significance of the above calculation is that, for the Thirring model at least, we have shown that the radiation emitted from a black hole is of a *thermal* equilibrium nature even in the presence of interactions. Thus the central conjecture of the Hawking phenomenon—that black holes are intrinsically thermal—is upheld. Previously Gibbons and Perry[9] argued, using perturbation theory, that this would be true for a general interacting system. In our calculation it has not been necessary to use perturbation theory: We have obtained an *exact* result.

IV. THE SPECTRUM

In a self-interacting quantum field theory there is no reason to suppose that the thermal equilibrium spectrum will have the Planck form. Deo and Kumar[1] attempted to evaluate the thermal spectrum for the Thirring model, but their result is incorrect for reasons given in the preceding section. What is more, the physical interpretation of their results is also incorrect for reasons to be given shortly. We restrict our treatment to Minkowski spacetime.

The interacting field components ϕ_1 and ϕ_2 are assumed to be equal to the free-field components, ψ_1 and ψ_2, at $t=0$. The free field can be expanded in standard exponential mode solutions

$$\psi_{1,2}(t,z)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}dp\left[\theta(\pm p)a^*(-p)e^{i(pz+Et)}+\theta(\mp p)b(p)e^{i(pz-Et)}\right], \tag{4.1}$$

which may be inverted at $t=0$ to obtain $b(p)$. Then multiplying by e^{iHt} on the right and e^{-iHt} on the left (H being the total Hamiltonian) one obtains the operators b at time t in terms of the interacting fields at time t:

$$b(p,t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}dz\,e^{-ipz}\left[\theta(p)\phi_2(t,z)+\theta(-p)\phi_1(t,z)\right]. \tag{4.2}$$

We wish to calculate the number of particles with energy $|p|$ in the thermal spectrum. The most convenient way of doing this is to look at the expectation value for the number of particles (associated with field quantization in the standard Minkowski coordinates) present in the Kruskal vacuum $|\bar{0}\rangle$ of a quantum black hole. We work only in the flat spacetime asymptotic region, where the Schwarzschild coordinates coincide with Minkowski coordinates. We follow Deo and Kumar's treatment but use the correct thermal two-point function. We thus write

$$\langle N_p\rangle \equiv\langle\bar{0}|b^*(p,t)b(p,t)|\bar{0}\rangle =\frac{1}{2\pi}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dz'\left[\theta(p)\langle\bar{0}|\phi_2^*(t,z')\phi_2(t,z)|\bar{0}\rangle+\theta(-p)\langle\bar{0}|\phi_1^*(t,z')\phi_1(t,z)|\bar{0}\rangle\right]e^{-ip(z-z')}. \tag{4.3}$$

Using the equal-time anticommutation relations, Eq. (4.3) may be rewritten

$$\langle N_p\rangle=\frac{VZ^{-1}}{2\pi}-\frac{1}{2\pi}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dz'\theta(p)\langle\bar{0}|\phi_2(t,z)\phi_2^*(t,z')|\bar{0}\rangle e^{-ip(z-z')}+\text{negative-momentum term}, \tag{4.4}$$

where Z is the field renormalization constant and V is the volume of space. We need only consider the positive-momentum term because by symmetry the negative-momentum term will contribute an equal quantity to $\langle N_p\rangle$. One sees from Eq. (4.4) that $\langle N_p\rangle$ may be evaluated by knowledge of the Kruskal vacuum two-point function $\langle\bar{0}|\phi_2\phi_2^*|\bar{0}\rangle$, considered in Sec. III, but evaluated at equal *Schwarzschild* time t.

The easiest way to evaluate (4.4) is to take the double Fourier transform of the two-point function. Following Klaiber, one obtains using null coordinates

$$\langle\bar{0}|\phi_2(t,z)\phi_2^*(t',z')|\bar{0}\rangle=\frac{1}{16M\pi}e^{-(u+u')/8M}\int_{-\infty}^{\infty}dk^u\int_{-\infty}^{\infty}dk^vF_2^{\overline{m}}(k)e^{-i(k^u\Delta\bar{v}+k^v\Delta\bar{u})}, \tag{4.5}$$

where the Fourier transform

$$F_2^{\bar{m}}(k)=[2\bar{m}\Gamma(c)\Gamma(c+1)]^{-1}\theta(k^u)\theta(k^v)\times\left(\frac{k^v}{2\bar{m}}\right)^c\left(\frac{k^u}{2\bar{m}}\right)^{c-1}. \tag{4.6}$$

A factor $(4M)^{-1}e^{-(u+u')/8M}$ in (4.5) comes from the parallel transport of the spinor and the transformation of $\gamma^{\bar{u}}$ involved in calculating (2.13) in Kruskal coordinates.

Now from the transformation equations (3.7) with $t=t'$, we have

$$\begin{aligned}\Delta\bar{u}&=-2e^{-t/4M}e^{\zeta/8M}\sinh(\Delta z/8M),\\ \Delta\bar{v}&=2e^{t/4M}e^{\zeta/8M}\sinh(\Delta z/8M),\end{aligned} \tag{4.7}$$

where $\zeta=z+z'$. We substitute (4.7) in (4.5), and change the integration variables to

$$\begin{aligned}q^u&=(4M)^{-1}e^{t/4M}e^{\zeta/8M}k^u,\\ q^v&=(4M)^{-1}e^{-t/4M}e^{\zeta/8M}k^v.\end{aligned} \tag{4.8}$$

This gives

$$\langle\bar{0}|\,\phi_2(t,z)\phi_2^*(t,z')\,|\bar{0}\rangle=\frac{1}{4\pi}\int_{-\infty}^{\infty}dq^u\int_{-\infty}^{\infty}dq^vF_2^m(q)\times e^{-4Mi(q^u-q^v)\sinh(\Delta z/8M)} \tag{4.9}$$

where $m=(4M^{-1})e^{\zeta/8M}\bar{m}$ as in Sec. III.

Finally, this expression is substituted into (4.4). We change the z,z' integration to ζ, Δz, and interchange with the q^u,q^v integration. This yields

$$\frac{\langle N_p\rangle}{V}=(2\pi)^{-1}\left(Z^{-1}-\frac{1}{4\pi}\int_{-\infty}^{\infty}dq^u\int_{-\infty}^{\infty}dq^v\int_{-\infty}^{\infty}d\Delta z\,F_2^m(q)e^{-4Mi(q^u-q^v)\sinh(\Delta z/8M)-ip\Delta z}\right). \tag{4.10}$$

In arriving at (4.10) we have performed the ζ integration to obtain a volume-of-space factor, $2V$ [to obtain this the limits of integration in (4.4) must be written as $-V/2$ to $V/2$ instead of $-\infty$ to ∞].

The evaluation of the integrals in (4.10) for $0<c<\frac{1}{4}$ is explained in Appendix B. The result is

$$\frac{\langle N_p\rangle}{V}=\frac{1}{2\pi}\left[Z^{-1}-\frac{(-1)^{-3c}}{2\pi}(4mM)^{-2c}e^{4\pi Mp}\frac{|\Gamma(\frac{1}{2}+4iMp+c)|^2}{\Gamma(1+2c)}\right]. \tag{4.11}$$

Note that setting $c=0$, hence $Z=1$, and using the identity $|\Gamma(\frac{1}{2}+iy)|^2=\pi\,\mathrm{sech}\,\pi y$ one recovers the Planck spectrum of $(2\pi)^{-1}(1+e^{8\pi Mp})^{-1}$ particles per unit volume at a Hawking temperature $T=(8\pi kM)^{-1}$.

Our result (4.11) has a form similar to that of Deo and Kumar.[11] However, they not only used an incorrect thermal Green's function in arriving at their result, they also used an integral representation valid only for $c=0$. It is clear from examination of (4.11), or the corresponding result of Deo and Kumar, that $\langle N_p\rangle$ calculated in this way *cannot* be interpreted as the physical number of particles for the interacting field and thus used in calculating the energy density as is done by Deo and Kumar. There are several reasons for this: (i) The result is complex (a problem avoided by Deo and Kumar). (ii) Both equations for $\langle N_p\rangle$ are ultraviolet divergent for $c\neq 0$: As $p\to\infty$ they blow up like p^{2c} (let alone our result depending on the renormalization constant Z, which is zero). (iii) The energy density calculated using $\langle N_p\rangle$ has a temperature dependence T^{2c+2} which, as we shall see in the next section, is incompatible with the known form of the conformal anomaly for the stress tensor.

It is not surprising that such a calculation of $\langle N_p\rangle$ for interacting fields should produce an ultraviolet-divergent result. Indeed in calculating the energy density or the entire stress tensor we should expect to have to use a suitable regularization procedure. In particular, in flat space we should expect to have to take normal products of the fields defined in such a way as to give finite results. Such normal products for the Thirring model have been discussed by Lowenstein[26] and Lowenstein and Schroer.[27]

Lowenstein[26] lays down several criteria which should be satisfied by a normal product, and then describes one of the many possible such normal-product procedures. Unfortunately it is too complex a procedure to allow the calculation of the particle spectrum as described above. Rather, to illustrate the effect of using normal products, we shall calculate $\langle N_p\rangle$ using a normal-product prescription which satisfies all Lowenstein's criteria except that it is dependent on the splitting direction between x and x'. This should cause no problems here, however, as we are integrating over these variables.

We define the normal product by

$$N[\phi^*_{\alpha'}(x')\phi_{\alpha_2}(x)]=(-m^2\Delta u\Delta v)^{(a+b\gamma_1^5\gamma_2^5)/4\pi}\phi^*_{\alpha_1}(x')\phi_{\alpha_2}(x)-\langle 0|\psi^*_{\alpha_1}(x')\psi_{\alpha_2}(x)|0\rangle\,,$$

$$\gamma_i^5=\begin{cases}-1, & \alpha_i=1\\ +1, & \alpha_i=2\end{cases}\quad i=1,2\,.$$

Then we see that if we insert the N operator into the integrand of (4.3) and take equal times $t=t'$ the only effect on (4.4) is to remove the troublesome Z^{-1} factor [note $Z=\lim_{\epsilon\to\infty}(-m^2\epsilon^2)^{(a+b)/4\pi}$], and to introduce a factor $(-m^2\Delta u\Delta v)^c=(-m\Delta z)^{2c}$ into the integrand.

Introducing these changes into (4.10) and performing the q^u, q^v integrations leaves the remaining Δz integral which can be expressed as

$$\frac{1}{2\pi}\left[1+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{y}{\sinh y}\right)^{2c}\operatorname{csch}(y+i\epsilon)e^{-8\pi i Mpy}dy\right]. \tag{4.13}$$

In the free-field case ($c=0$) the integral can be evaluated by closing the integration contour in the lower half y plane ($p>0$). The integrand has poles at $y=-i\epsilon-n\pi i$ with corresponding residues $(-1)^n\exp(-8n\pi Mp)$, $n=0,1,2,\ldots$. The result is the usual Planck spectrum. Unfortunately it is not possible to evaluate the integral in (4.13) for $c\neq 0$ in terms of elementary functions. However, it is easy to see that the additional factor from (4.12) has cured the ultraviolet divergence in the spectrum. Simply note that the additional factor $(-m\Delta z)^{2c}$ which comes from (4.12) appears in (4.13) as a factor of $(-8\pi mMy)^{2c}$ (using $y=\Delta z/8\pi M$). Thus, formally, expression (4.13) may be recovered from the un-normal-ordered expression by operating on the latter with $(-1)^{3c}m^{2c}d^{2c}/dp^{2c}$, and replacing the renormalization constant Z by 1. When these operations are carried out on the asymptotic form, $p\to\infty$, of (4.11), namely

$$\frac{1}{2\pi}\left[Z^{-1}-\frac{(-1)^{3c}}{\Gamma(1+2c)}\left(\frac{p}{m}\right)^{2c}\right], \tag{4.14}$$

the result is zero. The ultraviolet divergence has thus been replaced by reasonable asymptotic behavior.

We have computed the integral in (4.14) numerically. Writing the integral as a contour integral, (4.14) can be recast using standard techniques of complex analysis in the form of a principal-value integral, which finally allows us to write

$$\frac{\langle N_p\rangle}{V}=\frac{1}{2\pi}\left[\frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\left(\frac{y}{\sinh y}\right)^{2c}\frac{\sin(8\pi Mpy)}{\sinh y}dy\right], \tag{4.15}$$

where a useful check is once again to recover the Planck spectrum by calculating (4.15) exactly in the free-field limit. The results of numerical integration of (4.15) for the three values $c=0$ (free field), $c=0.1$, and $c=1.0$ are plotted in Fig. 1. All three graphs are very similar in shape as compared with the very different graphs obtained by Deo and Kumar for the cases $c=0$ and $c=0.1$.

It is possible to evaluate (4.15) as an infinite sum of known functions, by expanding $(\sinh y)^{-1}$ in powers of e^{-2y} and integrating term by term. The result is

$$\frac{\langle N_p\rangle}{V}=\frac{1}{2\pi}\left\{\frac{1}{2}-\left(\frac{2}{\pi}\right)^{(2c+1)}\sum_{n=0}^{\infty}\frac{\Gamma(2c+1+n)}{n!}\left[64\pi^2M^2p^2+(2c+2n+1)^2\right]^{-(2c+1)/2}\right.$$
$$\left.\times\sin\left[(2c+1)\arctan\left(\frac{8\pi Mp}{2c+2n+1}\right)\right]\right\}. \tag{4.16}$$

Numerical summation of terms in this series gives a check on our numerical integration, although the series is unfortunately very slowly convergent.

As was pointed out earlier, we have dealt with one of a large number of possible normal-product methods for the Thirring model. In view of this no real physical significance should be attached to $\langle N_p\rangle$ calculated above. Rather the quantity of

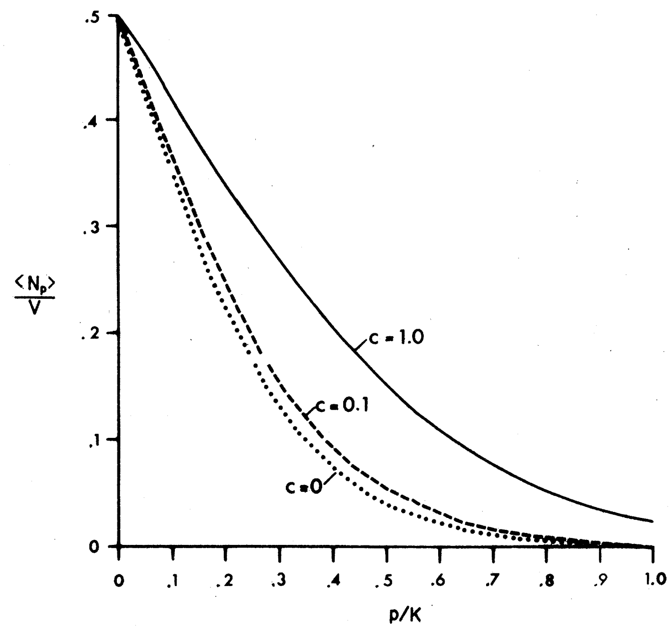

FIG. 1. The spectrum of Thirring fermions for three values of the coupling parameter, calculated numerically using the normal product given in Eq. (4.12).

physical interest is the properly quantized stress tensor, which we shall discuss in the next section. We notice, however, that if we calculate the energy density in the manner described by Deo and Kumar, but use the normal-product quantity $\langle N_p \rangle$ we obtain a result with temperature dependence T^2 which is compatible with the conformal anomaly to be discussed in the next section.

V. THE STRESS TENSOR

The stress tensor for a classical Thirring field may be obtained by functional differentiation of the Lagrangian density $\mathcal{L}$ using (2.2)

$$T_{\mu\nu}=\tfrac{1}{4}i[\bar{\phi},\gamma_\mu\nabla_\nu\phi]+\text{H. c.} -\lambda g_{\mu\nu}J^\sigma J_\sigma\,, \qquad (5.1)$$

where H. c. denotes Hermitian conjugate. In order to quantize this expression one must apply some regularization technique, and to this end, in flat space, Lowenstein and Schroer[27] apply the normal-product prescription given by Lowenstein in Ref. 26. However, they find that this does not give a correctly quantized stress tensor as $T_{\mu\nu}$ formed this way does not satisfy the commutation relations

$$\begin{aligned} i[P_\mu,\phi(x)]&=\partial_\mu\phi(x)\,,\\ i[L,\phi(x)]&=(x^0\partial_1+x^1\partial_0-\tfrac{1}{2}\tilde{\gamma}^5)\phi(x)\,,\\ i[D,\phi(x)]&=(x^0\partial_0+x^1\partial_1+\tfrac{1}{2}+g^2/4\pi^2)\phi(x)\,,\end{aligned} \qquad (5.2)$$

where the generators of translations, Lorentz transformations, and dilatations are defined formally by

$$\begin{aligned} P_\mu&=\int dz\,T_{0\mu}(t,z)\,,\\ L&=\int dz[tT_{01}(t,z)+zT_{00}(t,z)]\,,\\ D&=\int dz[tT_{00}(t,z)+zT_{01}(t,z)]\,,\end{aligned} \qquad (5.3)$$

respectively.

They are able to select another of the possible normal-product prescriptions which does properly quantize the stress tensor and find the following result, which is symmetric, traceless, and conserved:

$$T_{\mu\nu}=T^{(0)}_{\mu\nu}+\pi(g_{\mu\nu}g_{\nu\sigma}+\epsilon_{\mu\rho}\epsilon_{\nu\sigma})\,(\tfrac{1}{2}\xi^\rho\xi^\sigma-J^\rho\xi^\sigma)\,, \qquad (5.4)$$

where in terms of quantities described in Appendix A

$$\begin{aligned}\xi_\mu(x)&=j_\mu(x)-J_\mu(x)\\ &=\alpha\pi^{-1/2}Q\Delta_\mu(x)+\beta\pi^{-1/2}\tilde{Q}\tilde{\Delta}_\mu(x)\,,\end{aligned} \qquad (5.5)$$

and $T^{(0)}_{\mu\nu}=\frac{1}{2}i\!:\ \bar{\psi}\tilde{\gamma}_\mu\overleftrightarrow{\partial}_\nu\psi\!:$ is the free fermion stress tensor. Thus, apart from quantities which vanish when the infrared regulator mass goes to zero, the stress tensor in Minkowski space is identically the free-fermion stress tensor. Further evidence for this comes from the work of Dell'Antonio, Frishman, and Zwanziger,[24] and Callan, Dashen, and Sharp,[28] who show that the stress tensor is properly quantized if written in the Sugawara form

$$T_{\mu\nu}=\frac{1}{2\kappa}(J_\mu J_\nu+J_\nu J_\mu-g_{\mu\nu}J^\sigma J_\sigma)\,, \qquad (5.6)$$

where κ is a dimensionless constant defined by[29]

$$[J_0(t,z),J_1(t,z')]=i\kappa\partial[\delta(z-z')]/\partial z\,. \qquad (5.7)$$

Thus once again the free-fermion and Thirring stress tensors differ only by infrared terms which vanish as $m\to 0$. What is more, from Eq. (A4) we see that

$$T_{\mu\nu}=\tfrac{1}{2}(\partial_\mu j\,\partial_\nu j+\partial_\nu j\,\partial_\mu j-g_{\mu\nu}\partial^\sigma j\,\partial_\sigma j)\,, \qquad (5.8)$$

where $j(x)$ is a free scalar field. But (5.8) is the usual stress tensor for a free scalar field, demonstrating the equivalence of the Thirring model to a free scalar field theory. The equivalence of the free fermion and free scalar theories has also been demonstrated for curved space-times by Davies.[12]

We now present an argument for the equality of the Thirring and free-fermion stress tensors in curved spacetime. We show that consistency with curved-space renormalization results can only be obtained by assuming such an equality.

The expectation value of $T_{\mu\nu}$ as given by (5.1) is formally divergent (even for the free field) and must be regularized. This is accomplished in flat spacetime by normal-ordering the field products as in the work of Lowenstein and Schroer[27] described above. However, in curved spacetime this is no longer correct. There may be a nontrivial vacuum energy coming from the curvature which would be eliminated by normal-ordering (see, for example, DeWitt[30]). Instead, a *generally covariant* regularization procedure, such as point separation (see the work of Bunch, Christensen, Davies, and Fulling[13-16]) or dimensional regularization[31] must be used. These procedures all produce an *anomalous trace* in the renormalized stress tensor. That is, although the classical trace $T^\mu{}_\mu$ vanishes identically when the field equation is used, nevertheless the expectation value $g_{\mu\nu}\langle T^{\mu\nu}\rangle$ is nonzero in curved spacetime, and is given by a geometrical scalar. For free-fermion and boson fields it is found that

$$g_{\mu\nu}=\langle T^{\mu\nu}\rangle=-\frac{1}{24\pi}R\,. \qquad (5.9)$$

In the interacting case, such as the Thirring model, one expects (5.9) to be modified. However, it is hard to see how any modification other than an alteration of the coefficient $-1/24\pi$ is consistent with the requirement that the right-hand side be a geometrical scalar with dimensions $(\text{length})^{-2}$.

To pursue this we appeal to an important result of Christensen and Fulling.[17] They consider a completely general conserved stress tensor (whose classical trace vanishes) in the region outside a Schwarzschild type two-dimensional black hole. The covariant conservation equation

$$T^{\mu\nu}{}_{;\nu}=0 \qquad (5.10)$$

may be integrated explicitly in terms of the anomalous trace $T^\alpha{}_\alpha$, to yield an expression for the flux of Hawking radiation at great distance from the hole:

$$\text{energy flux}=\tfrac{1}{2}M^3\int_{2M}^{\infty}\frac{T^\alpha{}_\alpha(z)}{z^2}\,dz\,. \qquad (5.11)$$

Note that the flux evaluated at the *flat*-spacetime region depends on the form of the anomalous trace in the curved spacetime near the hole.

The importance of (5.11) is that one merely needs to know the energy density of thermal radiation in Minkowski space to fix the left-hand side. As was pointed out in the preceding section, the naive treatment leading to (4.11) (and in Deo and Kumar's work) gives the temperature dependence of this thermal energy density for the Thirring model as $T^{2+2c}\propto M^{-2-2c}$. This is clearly incompatible with (5.11) which gives an energy flux proportional to M^{-2} if $T^\alpha{}_\alpha\propto R=4M/z^3$. We have seen that application of a suitable normal-product prescription gives the correct temperature dependence.

The energy flux for large z may be calculated as the expectation of the component T_{uu} in the Kruskal vacuum $|\bar{0}\rangle$.[25] But we have seen that in the flat asymptotic region T_{uu} for the Thirring model and T_{uu} for the free fermion or free scalar fields are the same. Hence from (5.11) it follows that the coefficient of R in the anomalous trace $\langle T^\alpha{}_\alpha\rangle$ of the Thirring model must be the same as that of the free scalar and fermion fields, namely $-1/24\pi$. Now in the conformally trivial situation discussed here, a knowledge of the anomalous trace plus the quantum state determines the entire stress tensor uniquely, not only throughout the whole Schwarzschild spacetime, but for *all* two-dimensional curved spacetimes.[12] The result is

$$\langle T_{\mu\nu}\rangle=\theta_{\mu\nu}+\tfrac{1}{2}\alpha R g_{\mu\nu}\,, \qquad (5.12)$$

where

$$\theta_{uu}=2\alpha C^{1/2}\partial_u{}^2C^{-1/2},$$
$$\theta_{vv}=2\alpha C^{1/2}\partial_v{}^2C^{-1/2},$$
$$\theta_{uv}=\theta_{vu}=0,$$

and α is the anomaly coefficient: $g_{\mu\nu}\langle T^{\mu\nu}\rangle=\alpha R$. Thus the stress tensor for the Thirring model is equal to the free scalar and fermion stress tensors in any two-dimensional spacetime.

VI. PATH-INTEGRAL QUANTIZATION AND THE RENORMALIZATION GROUP

In this section we wish to consider the preceding results in the light of the important program of work of Drummond and Shore.[2-4] Their main result gives a general formula for the trace anomaly of a conformally invariant scalar field on an arbitrary four-dimensional compact manifold. We extend their result to conformal spinor fields in two dimensions and particularize to the Thirring model.

Our starting point is the path-integral generating functional for connected Green's functions (see, for example, Abers and Lee[32] or Freedman and Weinberg[33])

$$W[g_{\mu\nu},\eta,\bar\eta]=-i\ln\int[d\phi][d\bar\phi]\exp\left(i\int\{\mathcal{L}(x)[-g(x)]^{1/2}+\bar\eta(x)\phi(x)+\bar\phi(x)\eta(x)\}\right),\tag{6.1}$$

where W is a functional of the metric $g_{\mu\nu}$ and arbitrary anticommuting current η.[34]

We wish to consider dimensional regularization of (6.1) in which case the Lagrangian density $\mathcal{L}(x)$ for the Thirring model will be given by (2.2) but in n dimensions. With the n-dimensional metric given by

$$ds^2=C(x)\left[(dx^0)^2-\sum_{i=1}^{n-1}(dx^i)^2\right],\tag{6.2}$$

the γ matrices are given by

$$\gamma^\mu=C^{-1/2}\tilde\gamma^\mu,$$

$\tilde\gamma^\mu$ being the flat space Γ^μ of Delbourgo and Prasad,[35] who consider the Thirring model in flat space using dimensional regularization.

We shall not presuppose the lack of coupling-constant renormalization in the Thirring model but shall write the bare coupling constant in the Lagrangian as λ_0. This allows us to develop Drummond and Shore's argument for two-dimensional Lagrangians which may necessitate such a renormalization.

The energy-momentum tensor is now given in terms of the effective action $W=W[g_{\mu\nu},0,0]$ (Refs. 30 and 36),

$$\frac{\langle \text{out},0|T^{\mu\nu}|\text{in},0\rangle}{\langle\text{out},0|\text{in},0\rangle}=\frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g_{\mu\nu}},\tag{6.3}$$

which gives[30]

$$\langle\text{in},0|T^{\mu\nu}|\text{in},0\rangle=\frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g_{\mu\nu}}+\text{finite traceless terms},$$

and similarly for purely out states. Thus as far as the trace anomaly in either the in or out vacuum is concerned we need only calculate

$$\langle T^\mu{}_\mu(x)\rangle=\frac{2g_{\mu\nu}(x)}{[-g(x)]^{1/2}}\frac{\delta W}{\delta g_{\mu\nu}(x)}.\tag{6.4}$$

Let us now introduce a new metric $\bar g_{\mu\nu}$ conformally related to $g_{\mu\nu}$ by

$$\bar g_{\mu\nu}(x)=[\Omega(x)]^2g_{\mu\nu}(x).\tag{6.5}$$

We then have from (6.4)

$$\langle T^\mu{}_\mu(x)\rangle=\frac{2\bar g_{\mu\nu}(x)}{[-\bar g(x)]^{1/2}}\frac{\delta\bar W}{\delta\bar g_{\mu\nu}(x)}\bigg|_{\Omega=1}=\frac{\Omega(x)}{[-g(x)]^{1/2}}\frac{\delta\bar W}{\delta\Omega(x)}\bigg|_{\Omega=1}.\tag{6.6}$$

Now $\bar W$ in (6.6) is a functional of Ω and can be calculated as a sum over vacuum bubbles, the lowest-order term of which gives the free-field trace anomaly. The higher-order loops give an additional contribution W_I which in turn gives an addition $\langle T_I{}^\mu{}_\mu\rangle$ to the trace by way of (6.6).

For a diagram with p vertices we have

$$\bar W_I^{(p)}[\Omega]\propto\lambda_0^p\int\prod_{i=1}^{p}d^nx_i[\bar g(x_i)]^{1/2}[\bar\gamma^{\mu_i}]_{\alpha_i\beta_i}[\bar\gamma_{\mu_i}]_{\epsilon_i\xi_i}\times\prod_{1\le i<j\le p}\prod_{m=1}^{\lambda_{ij}}\bar S_{a_m}(x_i,x_j),\tag{6.7}$$

where $0\le\lambda_{ij}\le4$ is the number of lines from x_i to x_j and a_m labels a pair of spinor indices: $a_1=\beta_i\alpha_j$, $a_2=\beta_j\alpha_i$, $a_3=\xi_i\epsilon_j$, $a_4=\xi_j\epsilon_i$. $\bar S_{a_m}(x_i,x_j)$ is the free spinor propagator for the metric $\bar g_{\mu\nu}$, which satisfies

$$i\bar{\gamma}^{\mu}\bar{\nabla}_{\mu_{x_i}}\bar{S}(x_i,x_j)=-\bar{\delta}(x_i,x_j) \qquad (6.8)$$

with

$$\bar{\nabla}_{\mu}=\partial_{\mu}-\bar{\Gamma}_{\mu}\,, \qquad (6.9)$$

the spinor covariant derivative. The δ function is normalized such that

$$\int d^n x[\bar{g}(x)]^{1/2}\bar{\delta}(x_i,x_j)=1\,. \qquad (6.10)$$

We can easily show from (6.8)–(6.10) that $\bar{S}$ is related to the corresponding propagator S for the metric $g_{\mu\nu}$ by

$$\bar{S}(x_i,x_j)=\Omega^{(1-n)/2}(x_i)S(x_i,x_j)\Omega^{(1-n)/2}(x_j)\,. \qquad (6.11)$$

Also taking account of the fact that four lines end on each vertex we can write (6.7) as

$$\bar{W}_I^{(p)}[\Omega]\propto\int\prod_{i=1}^{p}d^n x_i[g(x_i)]^{1/2}\{\lambda_0[\Omega(x_i)]^{2-n}\}$$
$$\times\prod_{1\le i<j\le p}\prod_{m=1}^{\lambda_{ij}}S_{a_m}(x_i,x_j)\,. \qquad (6.12)$$

Note that by not allowing the $i=j$ terms in (6.7) and (6.12) we have taken account of the fact that dimensional regularization sets such terms equal to zero.

If we did not know before hand that the coupling constant in the Thirring model remains unrenormalized, following 't Hooft[37] (see also Collins and Macfarlane[38]) we should substitute an expansion for λ_0 in terms of a renormalized coupling constant λ and a mass μ:

$$\lambda_0=\mu^{2-n}\hat{\lambda}_0(\lambda)\equiv\mu^{2-n}\left[\lambda+\sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda)}{(n-2)^\nu}\right]. \qquad (6.13)$$

From 't Hooft's discussion of scaling of μ we can write the factor $\lambda_0\Omega^{2-n}$ in (6.12) in terms of a spacetime-dependent coupling constant $\lambda(x)$ as

$$\lambda_0[\Omega(x)]^{2-n}=[\mu\Omega(x)]^{2-n}\hat{\lambda}_0(\lambda)$$
$$=\mu^{2-n}\hat{\lambda}_0(\lambda(x))\,, \qquad (6.14)$$

where $\lambda(x)=\lambda(\lambda,\Omega(x))$ and $\lambda(\lambda,1)=\lambda$.

We now write the interaction part of (6.6) as

$$[-g(x)]^{1/2}\langle T_I{}^{\mu}{}_{\mu}(x)\rangle=\left[\Omega(x)\frac{d\lambda(x)}{d\Omega(x)}\right]\frac{\delta\bar{W}_I}{\delta\lambda(x)}\bigg|_{\lambda(x)=\lambda}.$$

But the term in brackets is precisely what 't Hooft tells us how to calculate [Eq. (3.9) of Ref. 37]; it is

$$\Omega(x)\frac{d\lambda(x)}{d\Omega(x)}=\left(1-\lambda\frac{\partial}{\partial\lambda}\right)a_1(\lambda)=\beta(\lambda)\,, \qquad (6.15)$$

with $\beta(\lambda)$ being the usual renormalization-group equation notation.[38]

We thus obtain for conformal spinor fields in two dimensions the result obtained by Drummond and Shore[2] for conformal scalar fields in four dimensions:

$$[-g(x)]^{1/2}\langle T_I{}^{\mu}{}_{\mu}(x)\rangle=\beta(\lambda)\frac{\delta\bar{W}_I}{\delta\lambda(x)}\bigg|_{\lambda(x)=\lambda}.$$

Now without even using an explicit solution of the Thirring model it can be deduced that there is no coupling-constant renormalization.[5] Thus $\beta(\lambda)$ and accordingly $\langle T_I{}^{\mu}{}_{\mu}\rangle$ are zero and so the total anomalous trace for the Thirring model is equal to the free-field trace as was found in the preceding section by totally different means.

We finally note that the use of path-integral quantization and dimensional regularization not only gives a generally covariant regularization method, but also avoids the ambiguities involved in quantization using normal products.

VII. CONCLUSION

Despite the aspects of triviality which make the Thirring model soluble, it has provided some useful insights into the behavior of interacting quantum fields in curved spacetimes. Foremost among these is the discovery that black-hole emission of Thirring fermions is thermal in nature, thus verifying conjectures to this effect for general interacting fields.[9]

We have shown that one of the consequences of this triviality is the equality of the Thirring model and the free fermion and free scalar stress tensors for any two-dimensional spacetime. Indeed using the arguments of Drummond and Shore this has been traced to the lack of coupling-constant renormalization in the Thirring model, a result of some interest in itself. However, this equality is in other ways unfortunate as it does not shed any new light on how the introduction of interactions might affect the back reaction of the field on the metric in the more realistic four-dimensional realm.

It is clearly of considerable interest to study other field theories in curved spacetimes, both to see whether they continue to contrive to preserve the second law of thermodynamics and emit thermal black-hole radiation and to observe the effect of the backreaction on the metric via Einstein's equations.

Little progress has been made so far in performing calculations with less trivial interacting field theories in curved spacetime. The most sophisticated calculations to date are those of Drummond[2] to third order in perturbation theory of $\lambda\phi^3_{(6)}$ and $\lambda\phi^4_{(4)}$ in Euclidean de Sitter space using dimensional regularization. Unfortunately it seems unlikely that it will be possible to apply such regularization techniques to less specialized curved spacetimes, as it is difficult to see how a general four-dimensional manifold can be meaningfully extended to n-complex dimensions. This is a problem which arises even with free-field theories and is overcome in other covariant regularization schemes such as point separation,[13-16] or adiabatic regularization.[39-41] It is thus of some importance to extend these schemes to interacting field theories. Some progress in this direction has been made by Birrell and Ford[42] using point separation on $\lambda\phi^4$.

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APPENDIX A

Here we summarize Klaiber's[18] operator solution of Eq. (2.3) (all quantities being those for flat space). He finds

$$\phi(x) = e^{i\chi^+(x)}\psi(x)e^{i\chi^-(x)}, \tag{A1}$$

where $\psi(x)$ is the solution of the free-field equation and

$$\chi^\pm(x) = K^\pm(x) + \Omega^\pm(x) \tag{A2}$$

with

$$K^\pm(x) = \alpha J^\pm(x) + \beta\gamma^5\tilde{J}^\pm(x),$$

$$\Omega^\pm(x) = \pi^{1/2}(\alpha Q + \beta\gamma^5\tilde{Q})[\Delta^\pm(x) + \gamma^5\Delta^\pm(x)], \tag{A3}$$

$$J^\pm(x) = j^\pm(x) - \alpha Q\Delta^\pm(x) - \beta\tilde{Q}\tilde{\Delta}^\pm(x),$$

where α and β are free parameters related to a and b of Sec. II by $a = \alpha^2 - 2\pi^{1/2}\alpha$, $b = \beta^2 - 2\pi^{1/2}\beta$. The free-field current j_μ is given by

$$j_\mu(x) = :\bar{\psi}(x)\gamma_\mu\psi(x): = \pi^{-1/2}\partial_\mu[j^+(x) + j^-(x)] \tag{A4}$$

where $j^\pm(x)$ are the positive- and negative-frequency parts of an infrared regularized free scalar field. The pseudocurrent $\tilde{j}_\mu(x)$ is given by

$$\tilde{j}_\mu(x) = \epsilon_{\mu\nu}j^\nu = \pi^{-1/2}\partial_\mu[\tilde{j}^+(x) + \tilde{j}^-(x)]. \tag{A5}$$

The charges Q and $\tilde{Q}$ are given by

$$Q = \int_{-\infty}^{\infty} j_0(x)dz, \quad \tilde{Q} = \int_{-\infty}^{\infty}\tilde{j}_0(x)dz, \tag{A6}$$

while the functions $\Delta^\pm(x)$ are given by

$$\Delta^\mp(x) = \pm\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{dk^1}{|k^1|}\theta(m - |k^1|)(e^{\mp ikx} - 1), \tag{A7}$$

which vanish as $m \to 0$.

Finally the current of the interacting field (2.4) is given by

$$J_\mu(x) = \pi^{-1/2}\partial_\mu J(x). \tag{A8}$$

APPENDIX B

We show here how to obtain (4.11) from (4.10).

Inspection of the q^v integral reveals that the result is unchanged if the sign of q^v in the exponent is reversed and the integral is multiplied by $(-1)^{-c}$. We may then perform the Δz integration:

$$\int_{-\infty}^{\infty} d\Delta z \exp[-4Mi(q^u + q^v)\sinh(\Delta z/8\pi M) - ip\Delta z] = \frac{8\pi Mi}{\sinh 8\pi Mp}[J_{8Mpi}(4Mi(q^u + q^v)) - e^{8\pi Mp}J_{-8Mpi}(4Mi(q^u + q^v))]. \tag{B1}$$

The remaining integrals have the form

$$\int_{-\infty}^{\infty} dq^u\int_{-\infty}^{\infty} dq^v\theta(4Mq^u)\theta(4Mq^v)J_\nu(4Mi(q^u + q^v))(q^u)^{c-1}(q^v)^c. \tag{B2}$$

Changing variables to $w = q^u + q^v$, $y = q^v - q^u$, (B2) reduces to

$$\tfrac{1}{2}\int_0^{\infty} dw\, J_\nu(4Miw)\int_{-w}^{w} dy\left(\frac{w+y}{2}\right)^c\left(\frac{w-y}{2}\right)^{c-1} = 2^{2c}B(c+1, c)(4Mi)^{-2c-1}\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu + c)/\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu - c) \tag{B3}$$

so long as $0 < c < \frac{1}{4}$.

Using the result (B3), we may write (4.10) as

$$\frac{1}{2\pi}\left\{Z^{-1}+\frac{(-1)^{2c}}{2}\frac{(8mM)^{-2c}}{\Gamma(2c+1)}\operatorname{csch}8\pi Mp\left[\frac{\Gamma(\frac{1}{2}+4iMp+c)}{\Gamma(\frac{1}{2}+4iMp-c)}-e^{8\pi Mp}\frac{\Gamma(\frac{1}{2}-4iMp+c)}{\Gamma(\frac{1}{2}-4iMp-c)}\right]\right\}. \tag{B4}$$

Extracting a factor $|\Gamma(\frac{1}{2}+4iMp+c)|^2$ from the term in square brackets leaves a factor which can be written in terms of exponentials. After simplification (B4) then reduces to Eq. (4.11).

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